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Practical work

paper exercises

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Contents

1 *Confidence Levels for normal distribution*

1.1 Measurement precision of thermometers

A company produces clinical thermometers.

a) From testing a sample of thermometers it is observed that the results from different thermometers spread approximately according to a normal distribution with a sigma of 0.1 degree celsius. Estimate how many of 10000 produced thermometers will show a temperature which is

  I) more than 0.3 degree wrong? (Note: can be either too low or to high)

  II) more than +0.3 degree wrong?

  III) more than 0.4 degree wrong?

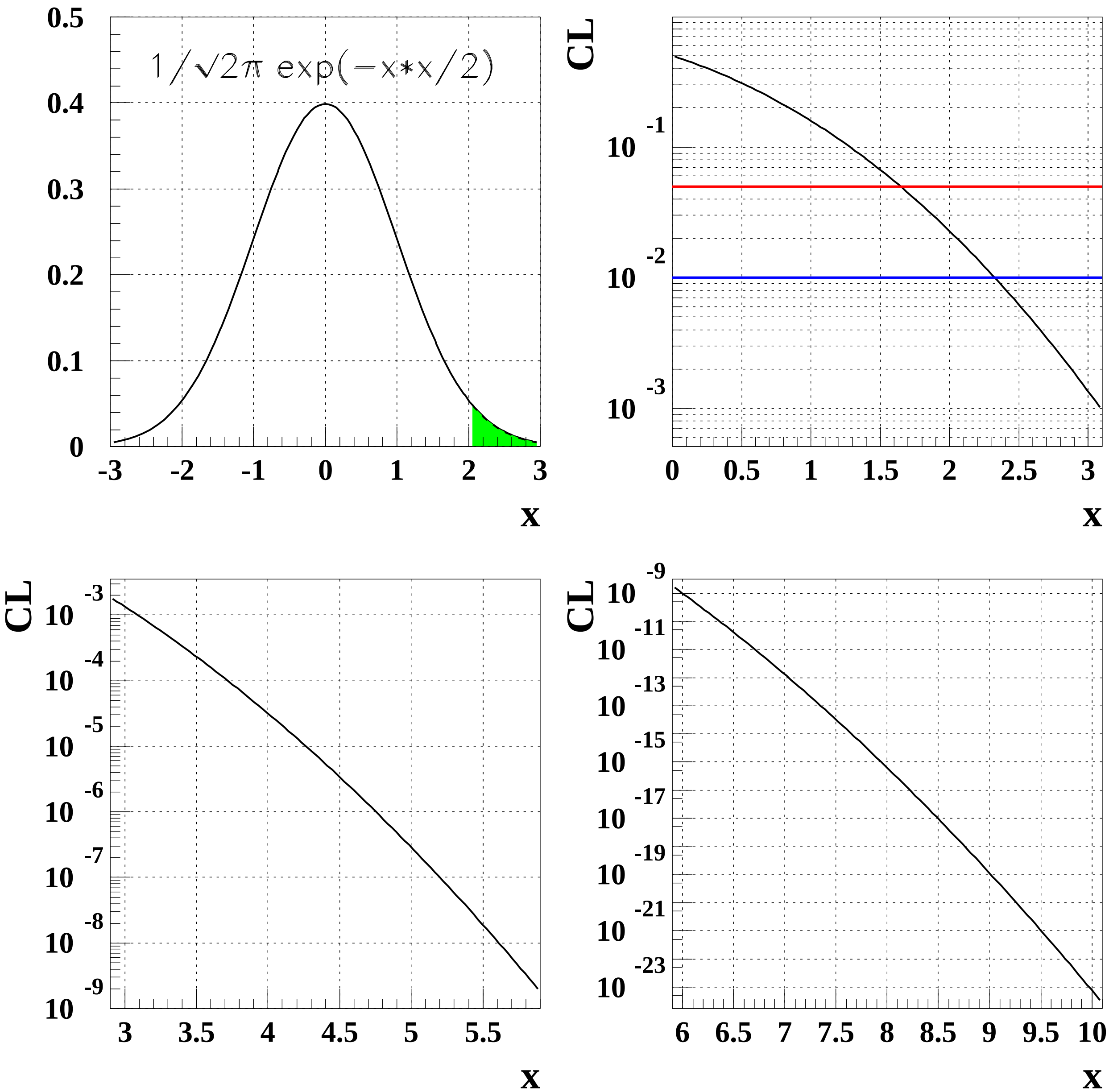
  IV) more than +0.4 degree wrong?

b) If one demands instead that less than 5% of the thermometers should be wrong by more than 0.1 degree - then to which precision (sigma) the thermometers should be calibrated?

Hint: Use the Confidence level curves for a gaussian function

$$CL(x) = \int_x^\infty dx' \frac{1}{\sqrt{2\pi}} e^{-x'^2/2}$$

Gauss Function one side confidence level vs x



Solution:

From reading the confidence level curves:

a) I) $0.3 = 3\,\sigma; \quad CL(3\sigma) = 2.7 \cdot 10^{-3}$
$\longrightarrow 10000{\cdot}0.0027 = 27$ thermometers are expected to be wrong like that

II) For single sided CL the number is just the half, ergo 13.5

III) more than 0.4 degree wrong?
$CL(4\sigma) = 6.3 \cdot 10^{-5} \longrightarrow 0.63$ thermometers

IV) more than +0.4 degree wrong?
$\longrightarrow$ 0.32 thermometers

b) 5% corresponds to 2σ, hence the σ should be $0.5 \cdot 0.1 = 0.05$ degrees.

1.2 Search for free quarks

An experiment was to look for quarks of charge $2e/3$, where e is the elementary charge. They should produce an ionisation of $4/9I_0$, where I_0 is the ionisation produced by a particle with the elementary charge. In an exposure of 10^6 cosmic particles, one track was measured to have $0.44I_0$.

$\longrightarrow$ Calculate the number of expected particles with true charge e, which would be measured with ionisation $I \leq 0.44\,I_0$ due a fluctuation of the ionisation measurement for the following two cases:

a) The ionisation estimates of the detector distribute as a Gauss function with $\sigma = 0.07\,I_0$ for all tracks

b) 99% of tracks with $\sigma = 0.07\,I_0$, while the rest with $\sigma = 0.14\,I_0$.

What are (your) conclusions for the possible discovery of free quarks?

Hint: Use the Confidence level curves for a gaussian function

Solution:

From reading the confidence level curves:

a) $0.44 I_0$ is 8σ away from the nominal value I_0 for a standard particle with the elementary charge $\rightarrow$ the chance for such a fluctuation or larger is

$$CL(8\sigma) = 10^{-15}.$$

The expected number of such tracks in a total sample of 1 M tracks is 10^{-9}.

b) for the 1% of tracks with $\sigma = 0.14 I_0$ $0.44 I_0$ is 4σ away corresponding to $CL(4\sigma) = 3 \cdot 10^{-5}$. The expected number of such tracks in a total sample of 1M tracks is thus $10^6 \cdot 0.01 \cdot 3 \cdot 10^{-5} = 0.3$

In case a) it seems like that a disovery was made, in case b) the event could be very well explained by a standard particle with fluctuating ionisation measurement.

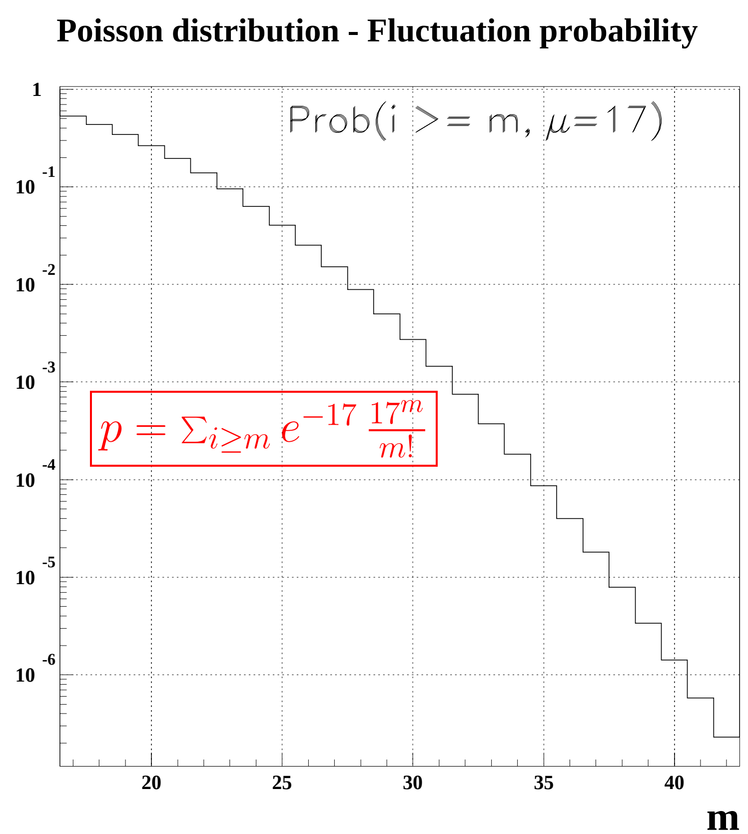
2 *Fluctuation probability for Poisson distribution*

2.1 Increased leukemia close to nuclear power plants

Researchers from Mainz (Maria Blettner et al) observed that in a 5 km surrounding of nuclear power plants 37 children contracted leukemia (in the years 1980 - 2003), while the statistical average in the population is 17. $\longrightarrow$ Determine the probability for a statistical fluctuation from 17 to ≥ 37:

a) Use the exact poisson probabilities as shown in the figure

b) Approximate the distribution by a gaussian with $\mu = 17$ and $\sigma = \sqrt{17}$. Use the CL curves for the gaussian to determine the fluctuation probability.



Solutions:

a) Simply reading off the figure: $p = 2 \cdot 10^{-5}$

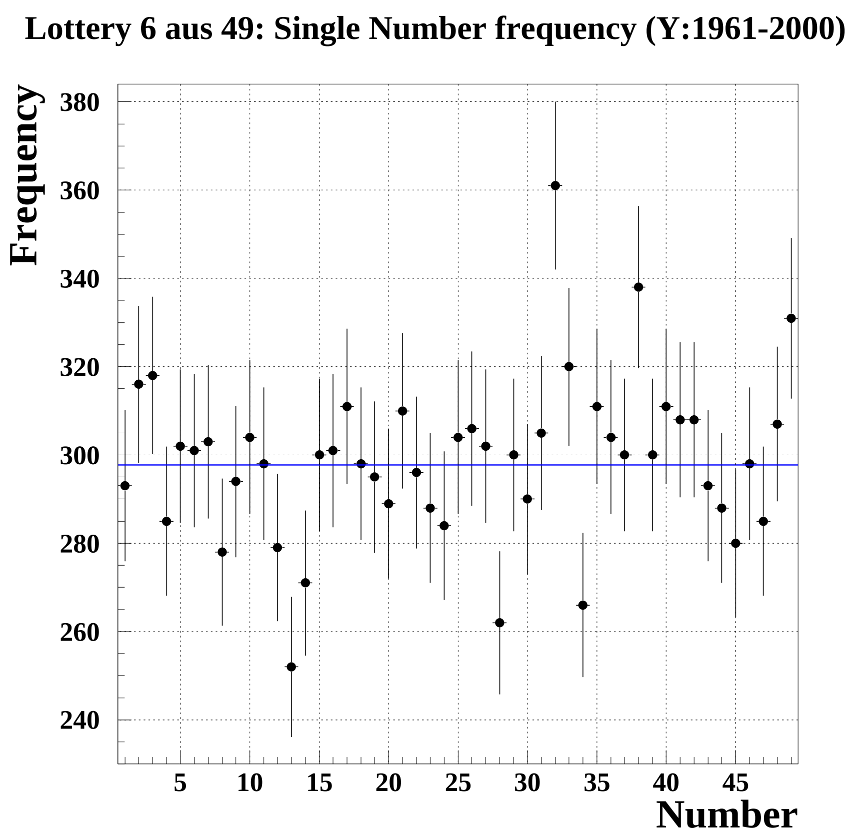
b) Deviation in number of σ: $(37 - 17)/\sqrt{17} = 4.85$
$\rightarrow CL = 6 \cdot 10^{-7}$

The difference between both estimates is due to the fact that the Poisson distribution has more tails towards larger numbers compared to the gaussian. However, in both cases, the fluctuation probability is very low such than one can conclude there is a significant increase in the cancer risk close to nuclear power plants. Further information:

- The results are vehemently disputed by advocats and opposers of nuclear power plants.
- The study gave also numbers for the number of all kind of cancer illnesses: 77 in the surrounding of nuclear power plants and 48 in the general population.

2.2 6 aus 49 Lottery (Streichaufgabe)

The frequency of drawing certain numbers in the german "6 aus 49 Lottery" (using 2088 draws from 1961-2000) is shown in the figure. The expectation value is 298. $\longrightarrow$ Check the probability (using gaussian approximation) for the observed largest upward and the largest downward fluctation to occur. Do you think everything is correct with this lottery?



Number	1	2	3	4	5	6	7	8	9	10
Freq	293	316	318	285	302	301	303	278	294	304
Number	11	12	13	14	15	16	17	18	19	20
Freq	298	279	252	271	300	301	311	298	295	289
Number	21	22	23	24	25	26	27	28	29	30
Freq	310	296	288	284	304	306	302	262	300	290
Number	31	32	33	34	35	36	37	38	39	40
Freq	305	361	320	266	311	304	300	338	300	311
Number	41	42	43	44	45	46	47	48	49	
Freq	308	308	293	288	280	298	285	307	331	

Solutions:

- Largest upward fluct.:
  $n = 32: \; (361 - 298)/\sqrt{298} = +3.64\sigma \rightarrow \; CL \sim 10^{-4}$
- Largest downwards fluct.:
  $n = 13: \; (252 - 298)/\sqrt{298} = -2.66\sigma \rightarrow \; CL \sim 0.004$

These fluctuations look significant. However, one must not forget that we have looked deliberately for the largest deviations. If one considers that there are 49 numbers to choose from... such single deviations are not unlikely. The $\chi^2 = \Sigma_{i=1,49} \frac{(f_i - 298)^2}{298} = 53$ for the number of degrees of freedom $ndf = 48$ is still reasonable (probability of $\sim 30\%$).

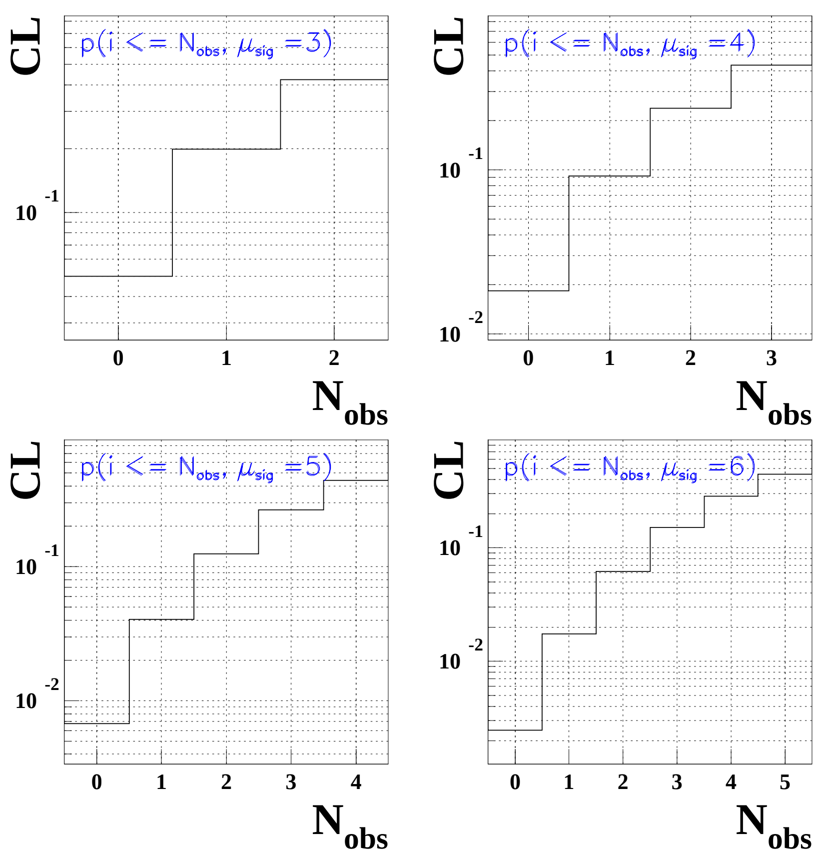
3 Limit determination for Poisson statistics

3.1 Particle production - basic limit determination

An experiment searches for the production of a new particle. After the final selection $N_{obs} = 2$ candidate events are observed. $\rightarrow$ Determine a 90% $C.L.$ upper limit on the expectation value μ of the underlying poisson distribution.

Instructions: The 90% upper limit value is given by the value μ for which the probability to observe N_{obs} or less events $p(\mu, N_{obs}) = \Sigma_{i \leq N_{obs}} e^{-\mu} \frac{\mu^i}{i!} = 10\%.$ For a selection of values μ these probabilities are shown in the figure below. From comparing the p values at $N_{obs} = 2$ try to estimate the μ for which $p(\mu, 2) = 0.1$.

Poisson distr. - Downward fluctuation probability



Solution:

From the figure:

$p(\mu = 3, 2) = 0.4$,

$p(\mu = 4, 2) = 0.22$,

$p(\mu = 5, 2) = 0.12$,

$p(\mu = 6, 2) = 0.06$

$\longrightarrow \mu \sim 5.3$

(exact solution, see e.g. PDG: $\mu = 5.32$)

3.2 Upper Limit for Signal + small background - frequentist approach

Most general the data consist of signal and background such that $\mu = \mu_{sig} + \mu_{bgr}$. Here μ_{sig} and μ_{bgr} are the Poisson parameters for signal and background respectively. Determine $90\%\ C.L.$ upper limits on μ_{sig} for the following cases with a given N_{obs} and known μ_{bgr}:

a) $\mu_{bgr} = 0,\ N_{obs} = 2$

b) $\mu_{bgr} = 1,\ N_{obs} = 2$

c) $\mu_{bgr} = 3,\ N_{obs} = 0$

Hint: Again the relevant formula to be used is

$$\boxed{p(\mu, N_{obs}) = \Sigma_{i \leq N_{obs}} e^{-\mu} \frac{\mu^i}{i!} = 10\%.}$$

to find a value for μ and then replacing $\mu = \mu_{sig} + \mu_{bgr}$.
Note: $p(\mu, N_{obs} = 0) = e^{-\mu}$.

Solution:

a) $\mu_{bgr} = 0$, $N_{obs} = 2$:
See previous exercise, $\mu_{sig} = 5.3$

b) $\mu_{bgr} = 1$, $N_{obs} = 2$:
$\mu_{sig} = 5.3 - \mu_{bgr} = 4.3$

c) $\mu_{bgr} = 3$, $N_{obs} = 0$: $p = e^{-(\mu_{sig}+3)} = 0.1$
$\rightarrow \mu_{sig}$ ought to be smaller than zero $\rightarrow \mu_{sig} = 0$.

3.3 Upper Limit for signal + small background - Modified frequentist approach

Determine (again) for the case $\mu_{bgr} = 3, N_{obs} = 0$ a 90% upper limit using the modified frequentist approach: $CL_s = CL(S+B)/CL(B) = 0.1$

Note: $CL(S+B)$ and $CL(B)$ are defined as

Hypothesis	CL
Background only	$CL(B) = p(\mu_{bgr}, N_{obs})$ $= \Sigma_{i \leq N_{obs}} e^{-\mu_{bgr}} \frac{\mu_{bgr}^i}{i!}$
Signal + Background	$CL(S+B) = p(\mu_{sig} + \mu_{bgr}, N_{obs})$ $= \Sigma_{i \leq N_{obs}} e^{-(\mu_{sig}+\mu_{bgr})} \frac{(\mu_{sig}+\mu_{bgr})^i}{i!}$

Solution:

$CL_s = CL(S+B)/CL(B) = e^{-(\mu_{sig}+\mu_{bgr})}/e^{-\mu_{bgr}} = e^{-\mu_{sig}} = 0.1 \rightarrow \mu_{sig} = -ln(0.1) = 2.3$

... as if there were no background!

(Reference: A.L. Read, (Oslo) CERN-OPEN-2000-205, Aug 2000.)

3.4 Upper Limit for particle negative yield measurement with gaussian errors - frequentist and Bayesian solution

An experiment "observes" after background subtraction a yield of $N = -2 \pm 1$ particles. $\rightarrow$ Determine an 90% upper limit μ_{lim} for the expectation value of events using

a) Frequentist approach: taking the results at face value
   Instruction: determine the 90% upper limit as usually for a measurement with gaussian error, i.e. from

$$CL = \int_{\mu_{lim}}^{\infty} dx' \frac{1}{\sqrt{2\pi}} e^{\frac{-(x'+2)^2}{2}} = 10\%$$

   Hint: The solution for μ_{lim} can be read off from the CL curves for a gaussian

b) Bayesian approach: the particle yields must be positive!
   Instruction: The limit μ_{lim} can be determined from

$$CL = \frac{\int_{\mu_{lim}}^{\infty} dx' \frac{1}{\sqrt{2\pi}} e^{\frac{-(x'+2)^2}{2}}}{\int_{0}^{\infty} dx' \frac{1}{\sqrt{2\pi}} e^{\frac{-(x'+2)^2}{2}}} = 10\%$$

   Hint: Both integrals can be looked up from the CL curves for a gaussian! For illustration see also the figures below

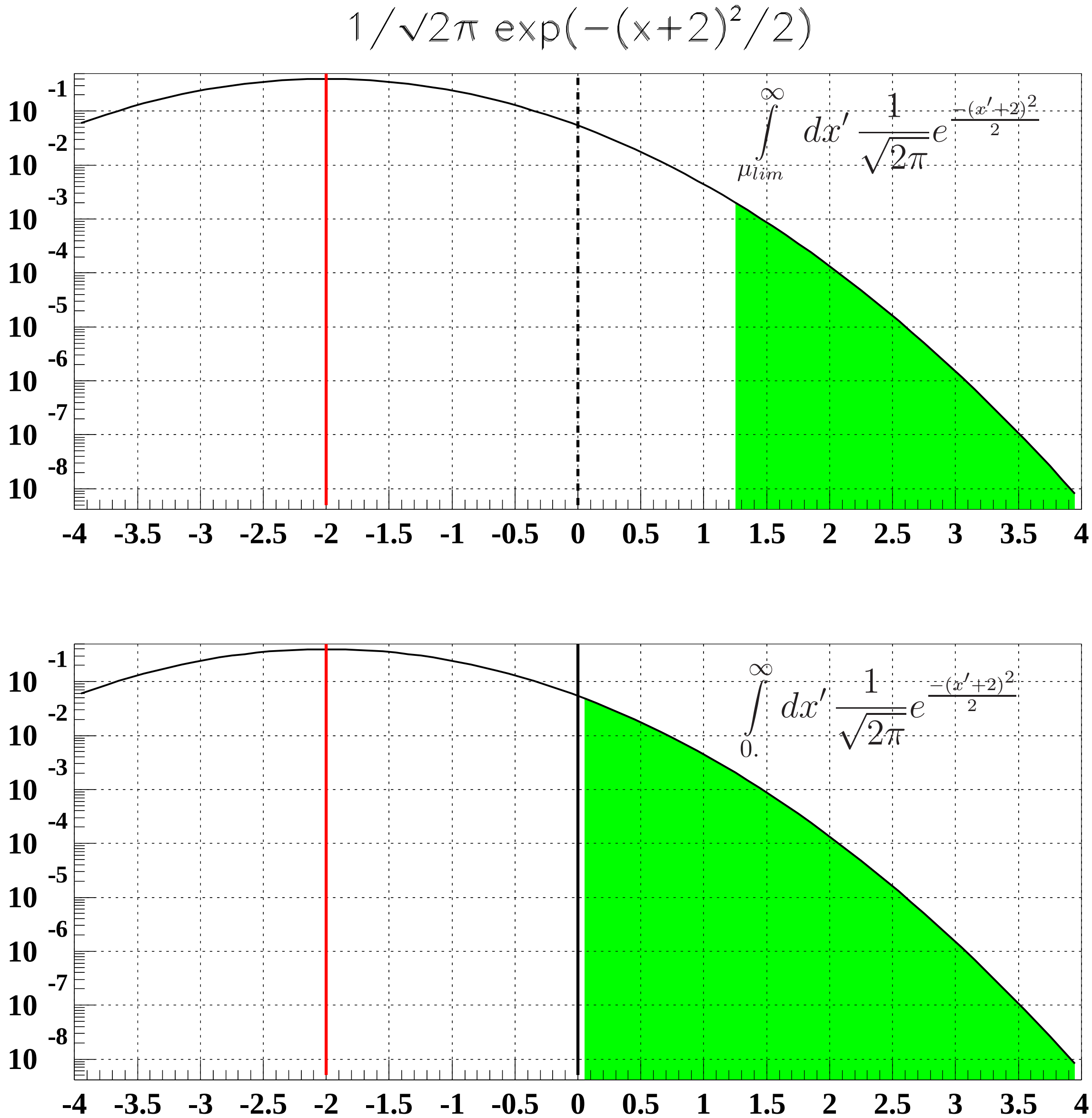


Figure 1: Gaussian with mean value -2 and width 1; the coloured areas show the integrals needed for the bayesian CL determination.

Solution:

a) Frequentist: from the CL curve:

$CL = 0.1 \leftrightarrow 1.28\sigma$

$\longrightarrow \mu_{lim} = -2 + 1.28 = -0.72$

b) Bayesian:

Renormalised total integral in physical area:

$\int\limits_{0.}^{\infty} dx' \frac{1}{\sqrt{2\pi}} e^{\frac{-(x'+2)^2}{2}} = CL(2) = 0.028$

Integral above limit:

$\longrightarrow \int\limits_{\mu_{lim}}^{\infty} dx' \frac{1}{\sqrt{2\pi}} e^{\frac{-(x'+2)^2}{2}} = 0.1 \cdot 0.028 = 0.0028$

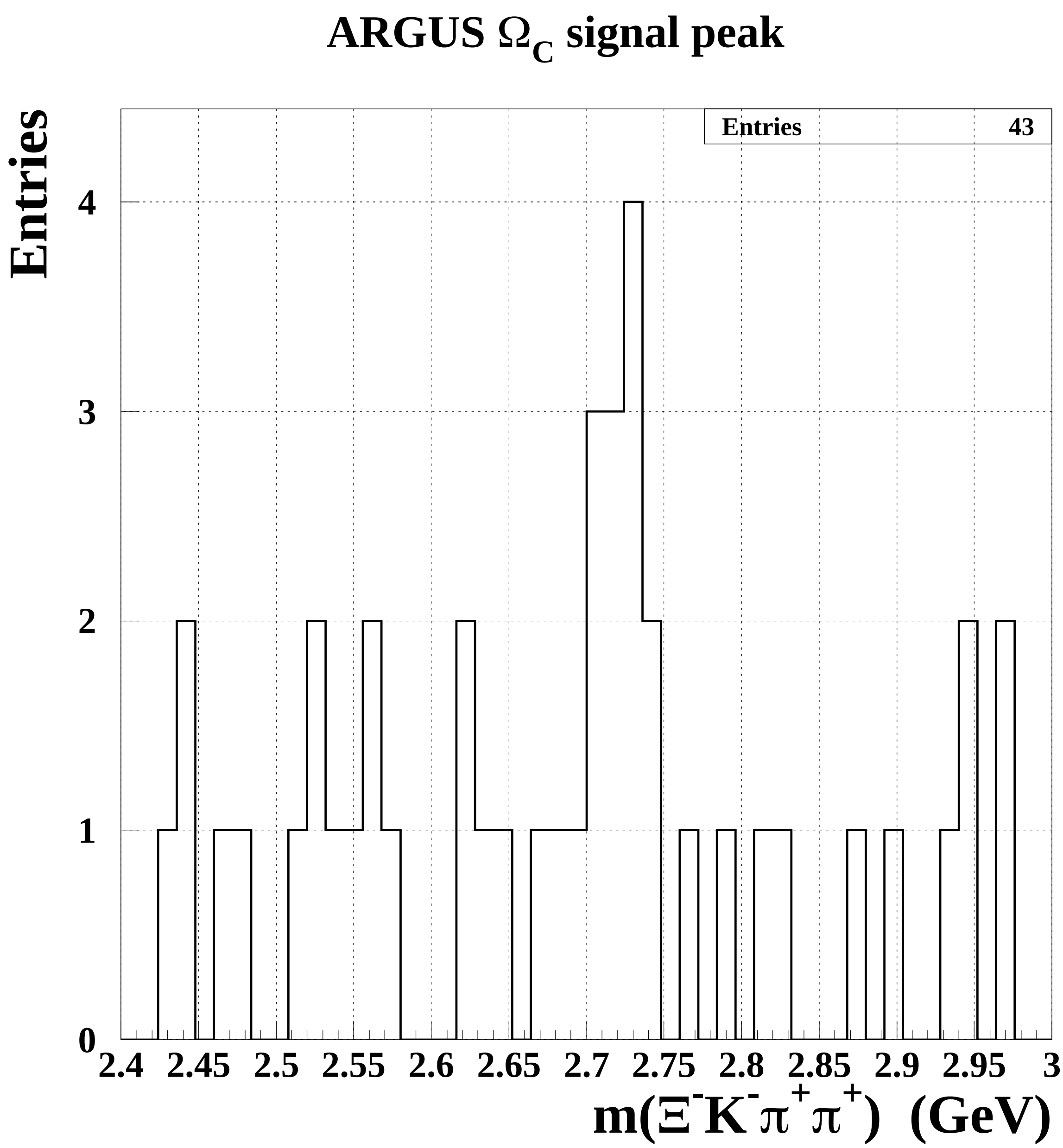
$CL = 0.0028 \leftrightarrow 2.75\sigma$

$\longrightarrow \mu_{lim} = -2 + 2.75 = 0.75$

4 *Signal discovery?*

4.1 Ω_c peak at ARGUS

The ARGUS e^+e^- experiment reported 1992 the observation of the charmed and doubly strange baryon Ω_c through its decay channel $\Xi^- K^- \pi^+ \pi^+$ (published in PL B288 367). The obtained mass spectrum is shown in the figure.



$\rightarrow$ Try to make your own assessment of the signal and its significance:

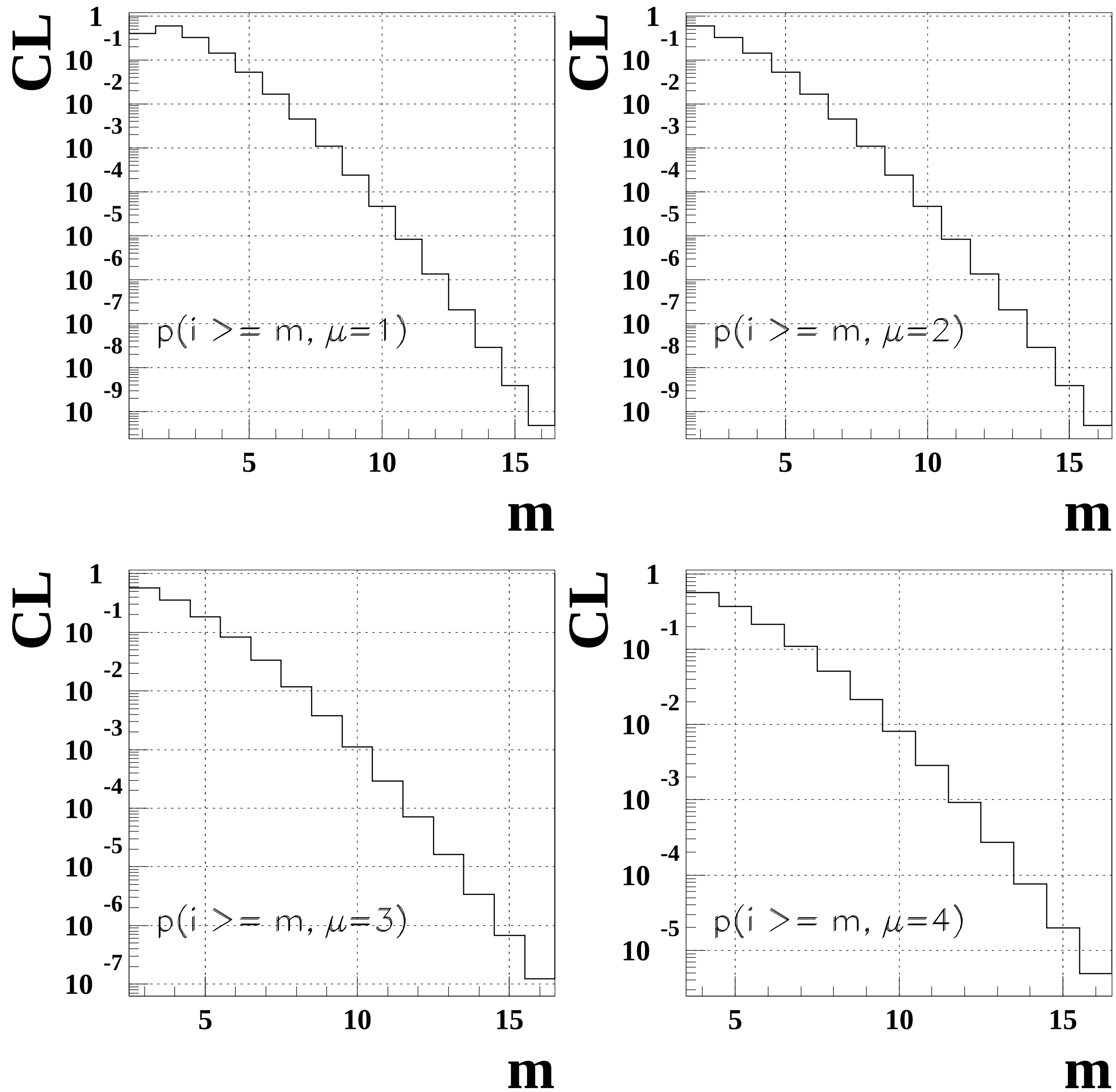
a) Fluctuation probability: Under the assumption there is only background with constant density:

   1. Estimate the average number of background events per mass bin (Note: the histogram contains 50 bins; the bin width is 12 MeV)
   2. Define a $\pm 2\sigma$ mass window around the peak (Note: the resolution is ~ 12 MeV, i.e. approximately the bin width)
   3. Count the total number of candidates $N_{cand,sig}$ in the $\pm 2\sigma$ region
   4. Estimate the number of expected background events μ_{bgr} in this region
   5. Estimate the probability for the poisson distribution to fluctuate from μ_{bgr} to $N_{cand,sig}$ or larger values (Probabilities for selected values μ are shown in the figure below)

b) Signal significance: Under signal + background hypothesis: Try to estimate the signal and its significance

1. Estimate the number of background events per bin from the entries in the sidebands of the peak
2. Estimate the number of background events μ_{bgr} in the $\pm 2\sigma$ region around the peak
3. Obtain the number $N_{sig} = N_{cand,sig} - \mu_{bgr}$, estimate an error $\sigma_{N_{sig}}$ and determine the signal significance $N_{sig}/\sigma_{N_{sig}}$.

Poisson distribution - Fluctuation probability



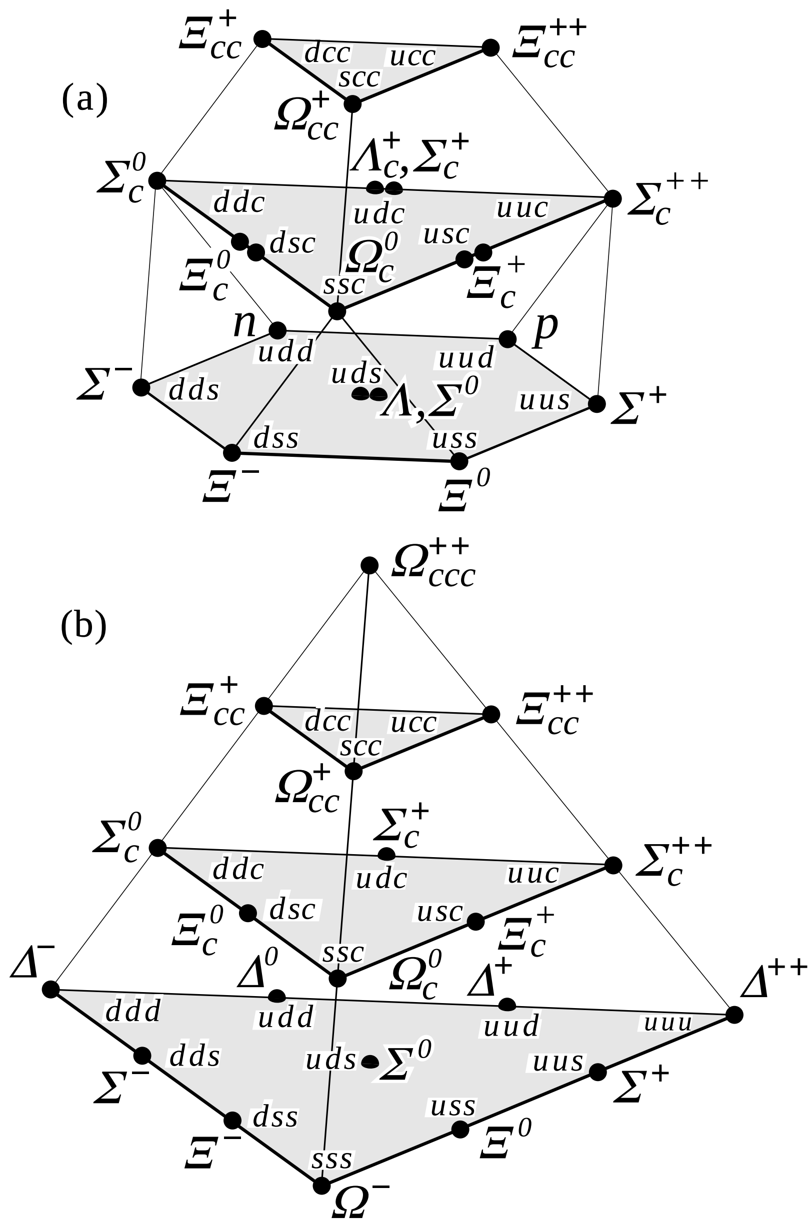
Solutions:

a) Fluctuation probability:

1. $\#bgr/bin = 43/50 = 0.86$
2. $\pm 2\sigma$ mass window: 2.7-2.748 GeV
3. $N_{cand,sig} = 12$
4. $4 \cdot 0.86 = 3.44$
5. Fluctuation probability: something between 10^{-4} and 10^{-3}. exact value $= 2 \cdot 10^{-4}$... Looks like a discovery!

b) 1. Sidebands: $(43 - 12)/46 = 0.67$ candidates/bin
2. $4 \cdot 0.67 = 2.7$
3. $N_{sig} = 12 - 2.7 = 9.3; \quad \sigma_{N_{sig}} \approx \sqrt{N_{cand,sig}} = \sqrt{12} = 3.46$
$\Rightarrow N_{sig}/\sigma_{N_{sig}} = 9.3/3.46 = 2.7$

Further information: The ω_c is supposed to be the css baryon ground state (see figure). It is still not known very well (PDG2006), its mass has been determined by CLEO2 to be $(2.6946 \pm 2.6 \pm 1.9)$ GeV.


(a)
Ξ_{cc}^{+}
Ξ_{cc}^{++}
dcc
ucc
scc
Ω_{cc}^{+}
Λ_c^+, Σ_c^+
Σ_c^0
Σ_c^{++}
ddc
udc
uuc
dsc
usc
Ξ_c^0
Ω_c^0
ssc
Ξ_c^+
n
p
udd
uud
Σ^-
dds
uds
Λ, Σ^0
uus
Σ^+
dss
uss
Ξ^-
Ξ^0
(b)
Ω_{ccc}^{++}
Ξ_{cc}^{+}
Ξ_{cc}^{++}
dcc
ucc
scc
Ω_{cc}^{+}
Σ_c^+
Σ_c^0
Σ_c^{++}
ddc
udc
uuc
dsc
usc
Ξ_c^0
Ξ_c^+
ssc
Ω_c^0
Δ^-
Δ^0
Δ^+
Δ^{++}
ddd
udd
uud
uuu
dds
uds
Σ^0
uus
Σ^-
Σ^+
dss
uss
Ξ^-
Ξ^0
sss
Ω^-

5 *Combination and compatibility of two measurements*

5.1 Direct CP violation ϵ'

The direct CP violation parameter $Re\left(\frac{\epsilon'}{\epsilon}\right)$ was measured by two different experiments to be (rounded numbers!)

$$Re\left(\frac{\epsilon'}{\epsilon}\right) = (7 \pm 6) \times 10^{-4} \text{ (E731)}$$

$$Re\left(\frac{\epsilon'}{\epsilon}\right) = (23 \pm 6) \times 10^{-4} \text{ (NA31)}$$

a) Determine from the two single measurements a combined result and error.

Hint: Weighted average $\hat{a}$ of two measurements a_i:

$\hat{a} = \frac{1}{g_1+g_2} \cdot (g_1 a_1 + g_2 a_2)$ with $g_i = 1/\sigma_i^2; \quad \sigma_{\hat{a}} = (g_1 + g_2)^{-0.5}$

b) Determine and compare the significances (= value/error) for the observation of direct CP violation for the single measurements and the combined one. Is there enough evidence to claim that direct CP violation was observed?

c) Estimate the compatibility of the two measurements from

$$\chi^2 = \sum_{i=1,2} \frac{(a_i - \hat{a})^2}{\sigma_i^2}$$

Express the compatibility from the probability to observe such a χ^2 or a larger one.

Hints: In the case of averaging two measurements the number of degrees of freedom for the χ^2 is $n = 1$. The requested probability can be looked up from the probability curves vs χ^2 (for different n) in figure 2.

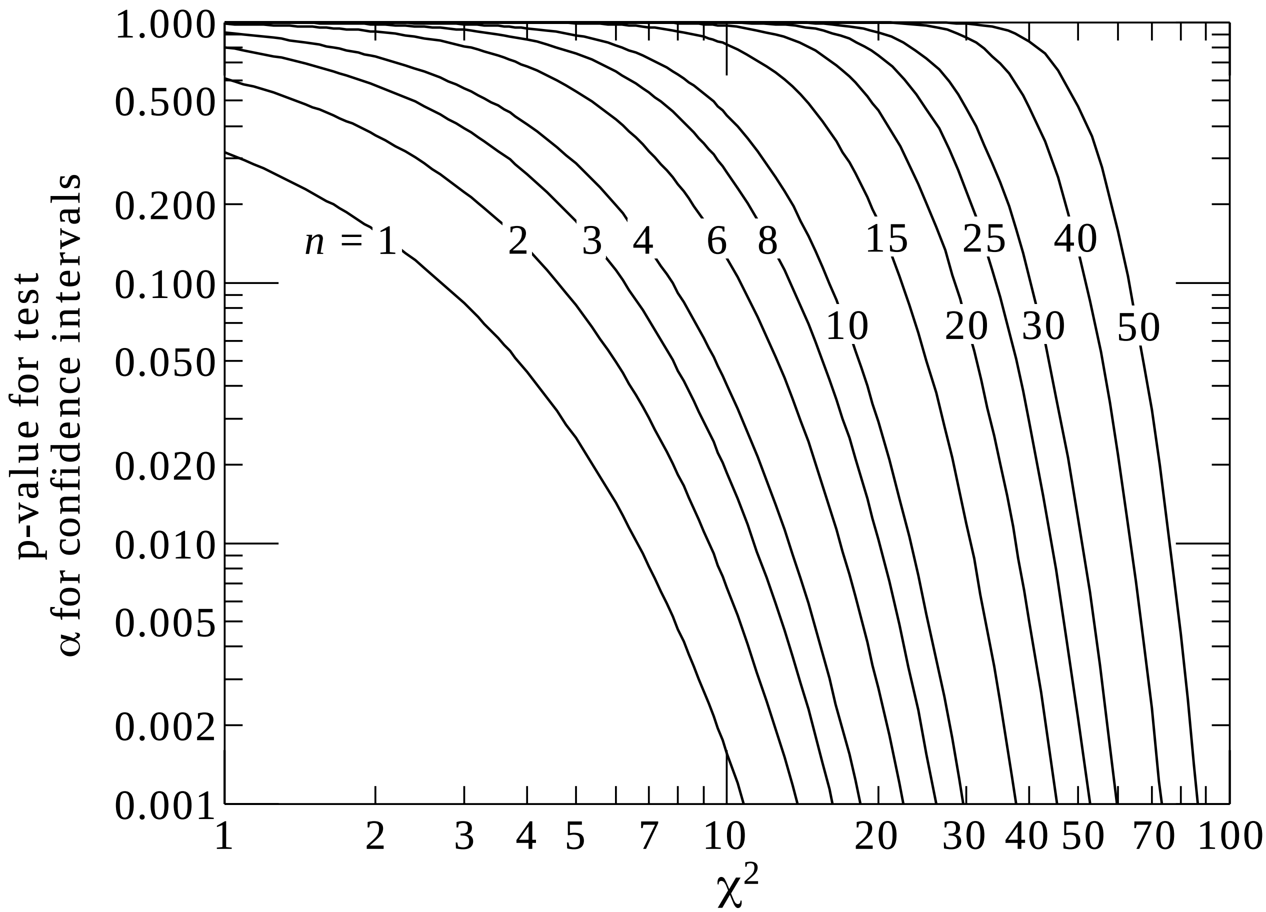


Figure 2: Probabilities to observe a χ^2 equal or larger than the given one for different degrees of freedom n (from the PDG).

Solutions:

a) $\hat{a} = \frac{1}{1/6^2+1/6^2} \cdot (1/6^2 \cdot 7 + 1/6^2 \cdot 23) = 15$

$\sigma_{\hat{a}} = (1/6^2 + 1/6^2)^{-0.5} = 4.2$

b) Significances:

E731: $7/6 = 1.2$

NA31: $23/6 = 3.8$

Combined: $15/4.2 = 3.6$

For NA31 alone there is $3.8\,\sigma$ evidence for direct CP violation, for the combined measurement 'only' $3.6\,\sigma$.

c) $\chi^2 = (7-15)^2/6^2 + (23-15)^2/6^2 = 3.55$ From the probability curves vs χ^2 for $n = 1$:

Probability = 0.05

Extra Information:

- Using the solution for $\hat{a}$ it can be easily shown that $\chi^2 = \frac{(a_1-a_2)^2}{\sigma_1^2+\sigma_2^2}$. It is evident that this should follow a gaussian distribution with width 1.
- The current status of the $Re\left(\frac{\epsilon'}{\epsilon}\right)$ measurements is shown in figure 3. (The figure containts also the exact numbers of the NA31 and E731 measurements).

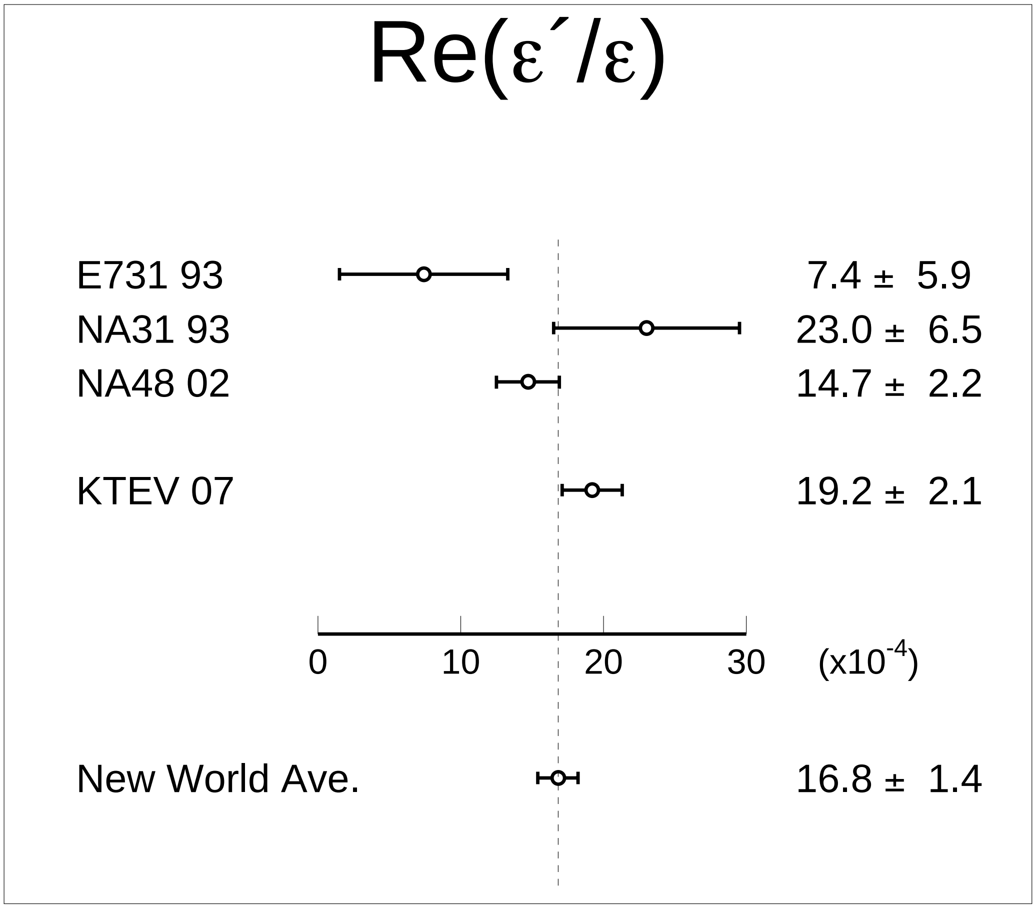


Figure 3: $Re\left(\frac{\epsilon'}{\epsilon}\right)$ results compilation in 2008

6 *Streichaufgabe: Max. Likelihood analytically*

6.1 Radioactive Decay

The probability density for radioactive decay of a certain substance is given by

$$p(t, \lambda) = \lambda e^{-\lambda t}$$

λ can be determined from a set of observed decay times, using the maximum likelihood method: Determine an estimate for λ for the case of

a) one single decay at time t_i by calculating

- $\omega = lnL = ln\, p(\lambda, t_i)$
- $d\omega/d\lambda$
- find the solution for λ from $d\omega/d\lambda = 0$

b) generalise to N decays. The Likelihoodfunction is now given by

$$L = \Pi_{i=1}^{N} \lambda e^{-\lambda t_i}$$

Determine for both cases a) and b) an estimate for the error of λ from

$$\sigma_{\hat{\lambda}} = \left(-d^2 ln(L)/d\lambda^2\right)^{-1/2}$$

(parabola approximation of lnL around the maximum)

Using instead χ^2 formulation:

Figure 5 shows the $\chi^2 = -2(ln(L) - ln(L_{max}))$ function for different number of radioactive decays. (coincidentally with minimas exactly at $\lambda = 1$). Determine graphically the $\pm$ errors of the estimated λ from the values of λ for which $\chi^2 = \chi^2_{min} + 1$ using the exact χ^2 curves and compare to using the parobala approximated curves.

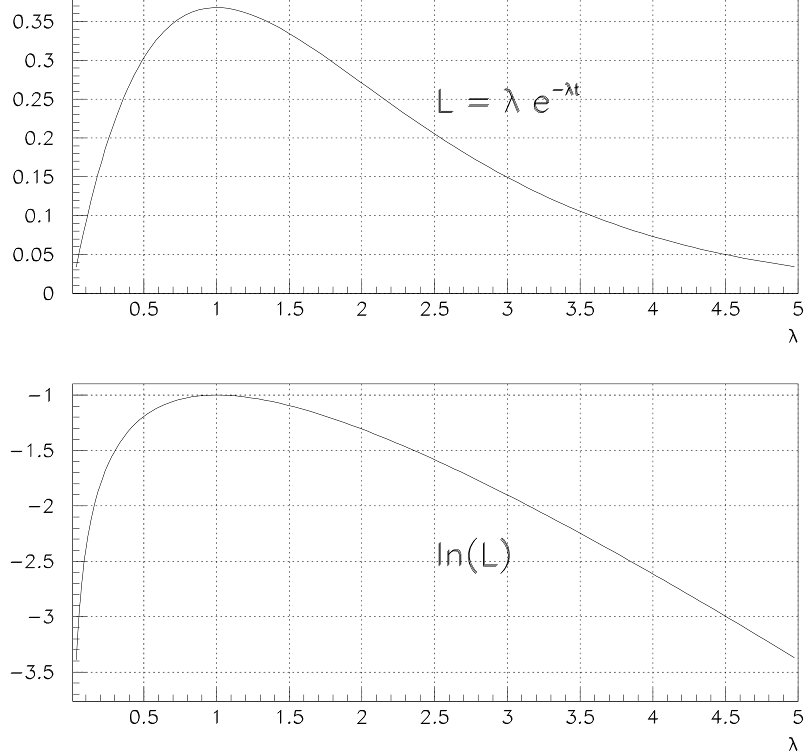


Figure 4: Likelihood function for single radioactive decay at time $t = 1$.

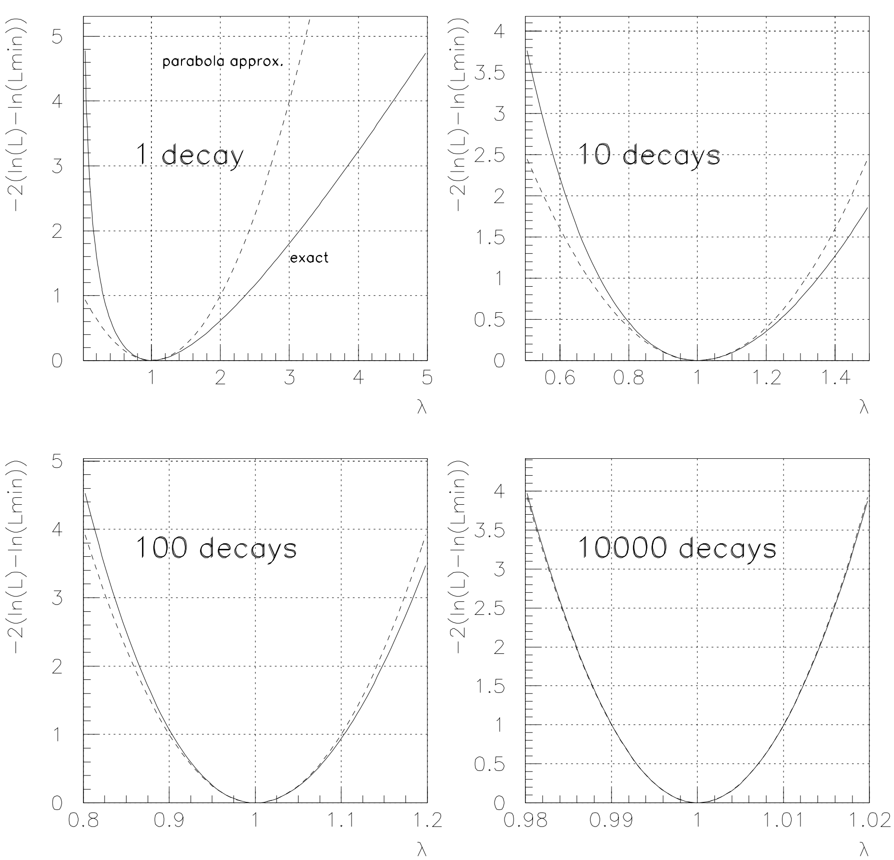


Figure 5: $\chi^2 = -2(ln(L) - ln(L_{max}))$ function for different number of radioactive decays (coincidentally with minimas exactly at $\lambda = 1$).

Solutions:

a) $-\ \omega = lnL = ln\, p(\lambda, t_i) = ln\,\lambda - \lambda t_i$

$-\ d\omega/d\lambda = \frac{1}{\lambda} - t_i$

$-\ d\omega/d\lambda = 0 \leftrightarrow \lambda = \frac{1}{t_i}$

b) $-\ \omega = lnL = \Sigma_{i=1,N}\, ln(L_i) = \Sigma_{i=1,N}\, ln\,\lambda - \lambda t_i = Nln(\lambda) - \lambda T$ with $T = \Sigma\, t_i$

$-\ d\omega/d\lambda = \frac{N}{\lambda} - T$

$-\ d\omega/d\lambda = 0 \leftrightarrow \lambda = \frac{N}{T}$

Error estimate $\sigma_{\hat{\lambda}} = (-d^2\omega/d\lambda^2)^{-1/2}$:

a) $\frac{d^2\omega}{d\lambda^2} = \frac{d}{d\lambda}(\frac{1}{\lambda} - t_i) = -\frac{1}{\lambda^2} \Rightarrow \sigma_{\hat{\lambda}} = \lambda = 1/t_i$

b) $\frac{d^2\omega}{d\lambda^2} = \frac{d}{d\lambda}(\frac{N}{\lambda} - T) = -\frac{N}{\lambda^2} \Rightarrow \sigma_{\hat{\lambda}} = \frac{\lambda}{\sqrt{N}} = \frac{\sqrt{N}}{T}$

Compare the $\pm$ errors graphically (see figure 5):

decays	$\chi^2 + 1$ left	$\chi^2 + 1$ right	parabola
1	0.6	1.4	1.
10	0.29	0.35	0.3
100	0.1	0.1	0.1